$$\boxed{\ell(\gamma) = \sum_{ij} w_{ij} \ln \frac{\gamma_i}{\gamma_i + \gamma_j} = \sum_{ij} w_{ij} \left(\ln \gamma_i - \ln (\gamma_i + \gamma_j) \right)}$$

$$\ln\left(\frac{x}{y}\right) \leq \frac{x}{y} - 1$$

$$1 - \ln y - \frac{x}{y} \leq - \ln x$$

These are detailed calculations for the MM paper of Hunter

~~→ Hunter is locking the ranking of one player so it's slower than normalizing after each iteration.~~

$$y = \gamma_i^{(k)} + \gamma_j^{(k)}$$
$$x = \gamma_i + \gamma_j$$

i beats j (n).

$$\ell(\gamma) \geq \sum_{ij} w_{ij} \left(\ln \gamma_i + 1 - \ln\left(\gamma_i^{(k)} + \gamma_j^{(k)}\right) - \frac{\gamma_i + \gamma_j}{\gamma_i^{(k)} + \gamma_j^{(k)}} \right)$$

$$\gamma_i^{(k+1)} = \arg\max \sum_{j} w_{ij} \left(\ln \gamma_i - \frac{\gamma_i}{\gamma_i^{(k)} + \gamma_j^{(k)}} \right) + \sum_j w_{ji} \left(- \frac{\gamma_i}{\gamma_j^{(k)} + \gamma_i^{(k)}} \right)$$

$$= \arg\max \underbrace{\left(\sum_j w_{ij} \right)}_{W_i} \ln \gamma_i - \sum_j \underbrace{(w_{ij} + w_{ji})}_{N_{ij}} \frac{\gamma_i}{\gamma_j^{(k)} + \gamma_i^{(k)}}$$

$$= \arg\max \; W_i \ln \gamma_i - \left(\sum_j \frac{N_{ij}}{\gamma_j^{(k)} + \gamma_i^{(k)}} \right) \gamma_i$$

$$f(x) = A \ln x - Bx$$

$$f'(x) = \frac{A}{x} - B \qquad f'(x) = 0 \text{ pour } \boxed{x = \frac{A}{B}}$$

$$- \ln (x + \gamma_j) \geq 1 - \ln(\gamma_i + \gamma_j) - \frac{x + \gamma_j}{\gamma_i + \gamma_j}$$

$$\boxed{\gamma_i^{(k+1)} = \frac{W_i}{\sum_j \frac{N_{ij}}{\gamma_j^{(k)} + \gamma_i^{(k)}}}}$$

θ_w : "~~home field advantage~~" "white advantage"

θ_d : ~~ties~~ draws

$\boxed{\text{i a les blancs}}$

$$P(i \text{ beats } j) = \frac{\theta_w \gamma_i}{\theta_w \gamma_i + \theta_d \gamma_j}$$

$$P(j \text{ beats } i) = \frac{\gamma_j}{\theta_w \theta_d \gamma_i + \gamma_j}$$

b_{ij}

$$P(i \text{ ties } j) = 1 - P(i \text{ beats } j) - P(j \text{ beats } i) = \cancel{\frac{(\theta_t\gamma_i + \theta_h\gamma_j)(\theta_h\gamma_i + \theta_t\gamma_j) - (\theta_t\gamma_i + \theta_h\gamma_j)\theta_h\gamma_i - (\theta_h\gamma_i + \theta_t\gamma_j)\theta_h\gamma_j}{(\theta_t\gamma_i + \theta_h\gamma_j)(\theta_h\gamma_i + \theta_t\gamma_j)}}$$

w_{ij} : nb de victoires de i (i a les blancs) sur j — d_{ij} : nb de nulles entre i et j

b_{ij} : — avec les noirs

$$\ell(\gamma) = \sum_{ij} w_{ij}\left(\ln \theta_w \gamma_i - \ln(\theta_w\gamma_i + \theta_d\gamma_j)\right) + b_{ij}\left(\ln \gamma_j - \ln(\theta_w\theta_d\gamma_i + \gamma_j)\right)$$

$$+ d_{ij} \ln$$

$$P(i \text{ ties } j) = \frac{(\theta_w\gamma_i + \theta_d\gamma_j)(\theta_w\theta_d\gamma_i + \gamma_j) - \theta_w\gamma_i(\theta_w\theta_d\gamma_i + \gamma_j) - \gamma_j(\theta_w\gamma_i + \theta_d\gamma_j)}{(\theta_w\gamma_i + \theta_d\gamma_j)(\theta_w\theta_d\gamma_i + \gamma_j)}$$

$$N = \gamma_i\gamma_j(\cancel{\theta_w} + \theta_w\theta_d^2 \cancel{-\theta_w} - \theta_w) = \cancel{\gamma_i\gamma_j\theta_d(\theta_w\theta_d - 1)}$$

$$= \gamma_i\gamma_j\theta_w(\theta_d^2 - 1)$$

{i a les blancs}

Here starts the derivation for combining draws with advantage of white

$$\ell(\vec{\gamma}, \theta_d, \theta_w) = \sum_{ij} w_{ij} \ln \frac{\theta_w \gamma_i}{\theta_w\gamma_i + \theta_d\gamma_j} + l_{ij} \ln \frac{\gamma_j}{\theta_d\theta_w\gamma_i + \gamma_j}$$

$$+ d_{ij} \ln \frac{\theta_w \gamma_i \gamma_j (\theta_d^2 - 1)}{(\theta_w\gamma_i + \theta_d\gamma_j)(\theta_d\theta_w\gamma_i + \gamma_j)}$$

~~victory of i against j~~

$\left.\begin{array}{l} w_{ij} \\ l_{ij} \\ d_{ij} \end{array}\right\}$ wins, losses, draws of i against j. i playing white.

$$= \sum_{ij} w_{ij}\Big[\ln \theta_w\gamma_i - \ln(\theta_w\gamma_i + \theta_d\gamma_j)\Big] + l_{ij}\Big[\ln\gamma_j - \ln(\theta_d\theta_w\gamma_i + \gamma_j)\Big]$$

$$+ d_{ij}\Big[\ln\gamma_i + \ln\gamma_j + \ln(\theta_w(\theta_d^2-1)) - \ln(\theta_w\gamma_i + \theta_d\gamma_j) - \ln(\theta_d\theta_w\gamma_i + \gamma_j)\Big]$$

$$Q_k(\gamma) = \sum_{ij} w_{ij}\left[\ln \theta_w\gamma_i + 1 - \ln(\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}) - \frac{\theta_w\gamma_i + \theta_d\gamma_j}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}}\right]$$

$$+ l_{ij}\left[\ln\gamma_j + 1 - \ln(\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}) - \frac{\theta_d\theta_w\gamma_i + \gamma_j}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}}\right]$$

$$+ d_{ij}\Big[\ln\gamma_i + \ln\gamma_j + \ln\theta_w + \ln(\theta_d^2 - 1) + 1 - \ln(\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}) - \frac{\theta_w\gamma_i + \theta_d\gamma_j}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}}$$

$$+ 1 - \ln(\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}) - \frac{\theta_d\theta_w\gamma_i + \gamma_j}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}}\Big]$$

$$\gamma_i^{(k+1)} = \arg\max \sum_{j} w_{ij}\left(\ln\gamma_i - \frac{\theta_w\gamma_i}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}}\right) + l_{ij}\left(-\frac{\theta_d\theta_w\gamma_i}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}}\right)$$

$$+ d_{ij}\left(\ln\gamma_i - \frac{\theta_w\gamma_i}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}} - \frac{\theta_d\theta_w\gamma_i}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}}\right)$$

$$+ w_{ji}\left(-\frac{\theta_d\gamma_i}{\theta_w^{(k)}\gamma_j^{(k)} + \theta_d^{(k)}\gamma_i^{(k)}}\right) + l_{ji}\left(\ln\gamma_i - \frac{\gamma_i}{\theta_d^{(k)}\theta_w^{(k)}\gamma_j^{(k)} + \gamma_i^{(k)}}\right)$$

$$+ d_{ji}\left(\ln\gamma_i - \frac{\theta_d\gamma_i}{\theta_w^{(k)}\gamma_j^{(k)} + \theta_d^{(k)}\gamma_i^{(k)}} - \frac{\gamma_i}{\theta_d^{(k)}\theta_w^{(k)}\gamma_j^{(k)} + \gamma_i^{(k)}}\right)$$

$$\gamma_i^{(k+1)} = \arg\max \Big(\overbrace{\sum_j w_{ij} + d_{ij} + l_{ji} + d_{ji}}^{A}\Big) \ln \gamma_i$$

$$- \Big(\underbrace{\sum_j \frac{(d_{ij}+w_{ij})\theta_w}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}} + \frac{(d_{ij}+l_{ij})\,\theta_d\theta_w}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)}+\gamma_j^{(k)}} + \frac{(d_{ji}+w_{ji})\,\theta_d}{\theta_w^{(k)}\gamma_j^{(k)}+\theta_d^{(k)}\gamma_i^{(k)}} + \frac{(d_{ji}+l_{ji})}{\theta_d^{(k)}\theta_w^{(k)}\gamma_j^{(k)}+\gamma_i^{(k)}}}_{B}\Big)\gamma_i$$

$$= \arg\max\ A \ln \gamma_i - B\gamma_i$$

$$\gamma_i^{(k+1)} = \frac{A}{B}$$

- Structure de données : Pour chaque joueur, liste des adversaires
  Pour chaque adversaire j : w_{ij} d_{ij} l_{ji} d_{ji}
  $d_{ij}+w_{ij}$, $d_{ji}+w_{ji}$, $w_{ij}+d_{ij}+l_{ji}+d_{ji}$

$w_{01} = 1$ $\qquad$ $w_{10} = 0$
$l_{01} = 0$ $\qquad$ $l_{10} = 0$
$d_{01} = 1$ $\qquad$ $d_{10} = 0$

$\boxed{\begin{array}{l}\gamma \text{ to Elo} \\ \text{Elo to } \gamma\end{array}}$

i : Player
j : Oponent
$\gamma_0^{(k)} = 1 \qquad \gamma_1^{(k)} = 1$

$$\gamma_0^{(k+1)} = \arg\max\ 2 \ln \gamma_0 - \left(\frac{2\theta_w}{\theta_w + \theta_d} + \frac{1\,\theta_d\theta_w}{\theta_d\theta_w + 1}\right)\gamma_0$$

$$\gamma_0^{(k+1)} = \frac{2}{\frac{2\theta_w}{\theta_w+\theta_d} + \frac{\theta_d\theta_w}{\theta_d\theta_w+1}}$$

$$\gamma_1^{(k+1)} = \arg\max\ 1 \ln \gamma_1 - \left(\frac{2\theta_d}{\theta_w+\theta_d} + \frac{1}{\theta_d\theta_w + 1}\right)$$

$$\theta_w^{(k+1)} = \arg\max \underbrace{\left(\sum_{ij} w_{ij} + d_{ij}\right)}_{\mathbb{C}} \ln \theta_w - \underbrace{\left(\sum_{ij} \frac{(w_{ij}+d_{ij})\gamma_i}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}} + \frac{(l_{ij}+d_{ij})\theta_d\gamma_i}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}}\right)}_{\mathbb{B}} \theta_w$$

$$\theta_w^{(k+1)} = \frac{\mathbb{C}}{\mathbb{D}}$$

$$\theta_d^{(k+1)} = \arg\max \underbrace{\left(\sum_{ij} d_{ij}\right)}_{E} \ln(\theta_d^2 - 1) - \underbrace{\left(\sum_{ij} \frac{(w_{ij}+d_{ij})\gamma_j}{\theta_w^{(k)}\gamma_i^{(k)} + \theta_d^{(k)}\gamma_j^{(k)}} + \frac{(l_{ij}+d_{ij})\theta_w\gamma_i}{\theta_d^{(k)}\theta_w^{(k)}\gamma_i^{(k)} + \gamma_j^{(k)}}\right)}_{F} \theta_d$$

- Ajouter (k)

$$f(x) = E \ln(x^2 - 1) - F x$$

$$f'(x) = \frac{2x}{x^2-1} E - F$$

$$f'(x) = 0 \qquad 2Ex = F(x^2 - 1) \qquad \cancel{Fx^2 - 2Ex - F = 0}$$

$$x^2 - 2\frac{E}{F}x - 1 = 0$$

$$\Delta = 4\left(\frac{E}{F}\right)^2 + 4$$

$$x = \frac{2\frac{E}{F} \pm \sqrt{\Delta}}{2} = \frac{E}{F} \pm \sqrt{\left(\frac{E}{F}\right)^2 + 1}$$

Paul Hunter

$\ell_\alpha \in \frac{1}{T}$

- Enlever les (k) dans les formules, pour alléger

- Matrice Hessienne (Hessian Matrix)

$$l(\vec{\gamma}, \theta_d, \theta_w) = \sum_{ij} w_{ij} \ln \frac{\theta_w \gamma_i}{\theta_w \gamma_i + \theta_d \gamma_j} + l_{ij} \ln \frac{\gamma_j}{\theta_d \theta_w \gamma_i + \gamma_j}$$

$$+ d_{ij} \ln \frac{\theta_w \gamma_i \gamma_j (\theta_d^2 - 1)}{(\theta_w \gamma_i + \theta_d \gamma_j)(\theta_d \theta_w \gamma_i + \gamma_j)}$$

$$\frac{dl}{d\gamma_p} = \frac{d}{d\gamma_p} \Bigg(\sum_j w_{pj} \left(\ln \gamma_p - \ln(\theta_w \gamma_p + \theta_d \gamma_j) \right)$$
$$- l_{pj} \ln (\theta_d \theta_w \gamma_p + \gamma_j)$$
$$+ d_{pj} \left(\ln \gamma_p - \ln(\theta_w \gamma_p + \theta_d \gamma_j) - \ln(\theta_d \theta_w \gamma_p + \gamma_j) \right)$$
$$+ \sum_i -w_{ip} \ln (\theta_w \gamma_i + \theta_d \gamma_p)$$
$$+ l_{ip} \left(\ln \gamma_p - \ln(\theta_d \theta_w \gamma_i + \gamma_p) \right)$$
$$+ d_{ip} \left(\ln \gamma_p - \ln(\theta_d \theta_w \gamma_i + \gamma_p) - \ln(\theta_w \gamma_i + \theta_d \gamma_p) \right) \Bigg)$$

$$= \frac{d}{d\gamma_p} \Bigg(\left(\sum_j w_{pj} + d_{pj} + l_{jp} + d_{jp} \right) \ln \gamma_p$$
$$- \sum_j (w_{pj} + d_{pj}) \ln (\theta_w \gamma_p + \theta_d \gamma_j)$$
$$- \sum_j (l_{pj} + d_{pj}) \ln (\theta_d \theta_w \gamma_p + \gamma_j)$$
$$- \sum_j (w_{jp} + d_{jp}) \ln (\theta_w \gamma_j + \theta_d \gamma_p)$$
$$- \sum_j (l_{jp} + d_{jp}) \ln (\theta_d \theta_w \gamma_j + \gamma_p) \Bigg)$$

$$= \sum_j w_{pj} + d_{pj} + l_{jp} + d_{jp}$$

$$- \sum_j (w_{pj} + d_{pj}) \frac{d(\ln(\theta_w \gamma_p + \theta_d \gamma_j))}{d \ln \gamma_p} + (l_{pj} + d_{pj}) \frac{d(\ln(\theta_d \theta_w \gamma_p + \gamma_j))}{d \ln \gamma_p}$$

$$+ (w_{jp} + d_{jp}) \frac{d(\ln(\theta_w \gamma_j + \theta_d \gamma_p))}{d \ln \gamma_p} + (l_{jp} + d_{jp}) \frac{d(\ln(\theta_d \theta_w \gamma_j + \gamma_p))}{d \ln \gamma_p}$$

$$\boxed{r_i = \ln \gamma_i \qquad \gamma_i = e^{r_i}}$$

$$\boxed{\gamma = 10^{\frac{r}{400}} \qquad r = 400 \log \gamma}$$

$$- \frac{d^2 \ell}{dr_p dr_p} = \sum_j (w_{pj} + d_{pj}) \frac{d^2 \ln(\theta_w \gamma_p + \theta_d \gamma_j)}{d r_p^2} + \dots$$

$p \neq q$

$$- \frac{d^2 \ell}{dr_p dr_q} = (w_{pq} + d_{pq}) \frac{d^2 \ln(\theta_w \gamma_p + \theta_d \gamma_q)}{dr_p \, dr_q} + (l_{pq} + d_{pq}) \frac{d^2 \ln(\theta_d \theta_w \gamma_p + \gamma_q)}{dr_p \, dr_q}$$

$$(w_{qp} + d_{qp}) \frac{d^2 \ln(\theta_w \gamma_q + \theta_d \gamma_p)}{dr_p \, dr_q} + (l_{qp} + d_{qp}) \frac{d^2 \ln(\theta_d \theta_w \gamma_q + \gamma_p)}{dr_p \, dr_q}$$

$$\frac{d^2 \ln(\alpha \gamma_p + \beta \gamma_q)}{dr_p \, dr_q} = ? \qquad \frac{d^2 \ln(\alpha \gamma_p + \beta \gamma_q)}{d r_p^2} = ?$$

$$\frac{d^2 \ln(\alpha e^{r_p} + \beta e^{r_q})}{dr_p \, dr_q} = \frac{d}{dr_q} \left(\frac{\alpha e^{r_p}}{\alpha e^{r_p} + \beta e^{r_q}} \right) = \cancel{\frac{d}{dr_q} \left(\frac{1}{1 + \frac{\beta}{\alpha} e^{r_q - r_p}} \right)}$$

$$= - \frac{\beta e^{r_q} \alpha e^{r_p}}{(\alpha e^{r_p} + \beta e^{r_q})^2} = \boxed{- \frac{\alpha \beta \gamma_p \gamma_q}{(\alpha \gamma_p + \beta \gamma_q)^2}}$$

termes non diagonaux

$$\frac{d^2 \ln(\alpha e^{r_p} + \beta e^{r_q})}{d r_p^2} = \frac{d}{dr_p} \left(\frac{\alpha e^{r_p}}{\alpha e^{r_p} + \beta e^{r_q}} \right) = \frac{\alpha e^{r_p} (\alpha e^{r_p} + \beta e^{r_q}) - \alpha e^{r_p} \alpha e^{r_p}}{(\alpha e^{r_p} + \beta e^{r_q})^2}$$

$$= \boxed{\frac{\alpha \beta \gamma_p \gamma_q}{(\alpha \gamma_p + \beta \gamma_q)^2}}$$

$$\sum \gamma_i = c^{te}$$

$$\frac{d^2 \ell}{dR_i^2} = \left(\frac{\ln 10}{400} \right)^2 \frac{d^2 \ell}{dr_i^2}$$

- ~~Quel est le joueur dont on fixe le classement ?~~
- ~~Etant donnés n-1 variables aléatoires $X_1, \dots, X_{n-1}$~~

~~$X_n = \sum_{i<n} \alpha_i X_i$~~

~~covariance ?~~

$Y_i = X_i - \frac{1}{n} \sum X_i$

$Y_n = 0 - \frac{1}{n} \sum X_i$

Si on a la covariance de n-1 joueurs, le n-ième ayant un classement fixe, comment obtenir la covariance des n ?

~~On a des~~ n-1 variables dont on connait la covariance.

$Y_1 = X_1$

$Y_{n-1} = X_{n-1}$

$X_n = -\sum_{i<n} X_i$

$C = -H^{-1}$

C_i : matrice de covariance des X_i

$Y = AX$

$$Y = AX \qquad A = \begin{array}{c} n-1 \\ \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ -1 & -1 & -1 \end{pmatrix} \end{array} \; n$$

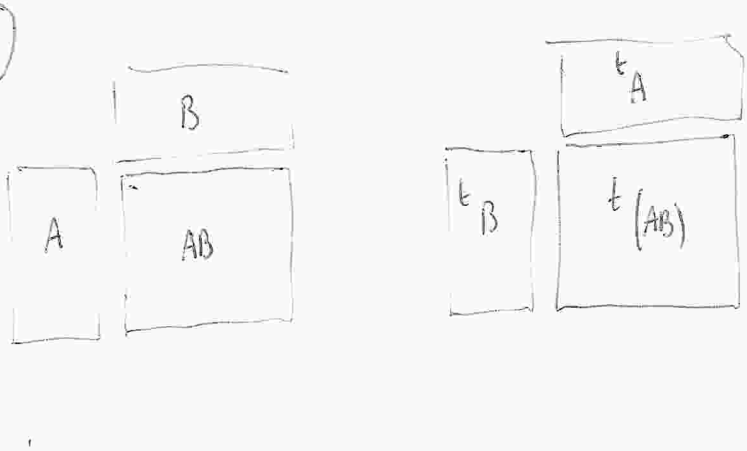
$$A = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ & \ddots & & \\ & & & 1-\frac{1}{n} \end{pmatrix}$$

$$\cancel{A = \begin{pmatrix} \frac{n-2}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ & & & \frac{n-2}{n-1} \\ \frac{-1}{n-1} & & \cdots & \frac{-1}{n-1} \end{pmatrix}}$$

$D = AC\,{}^tA$

Il ne reste plus qu'à programmer tout ça.

${}^t(AB) = {}^tB\,{}^tA$



$vm = {}^t(C\,{}^tA) = A\,{}^tC = AC$

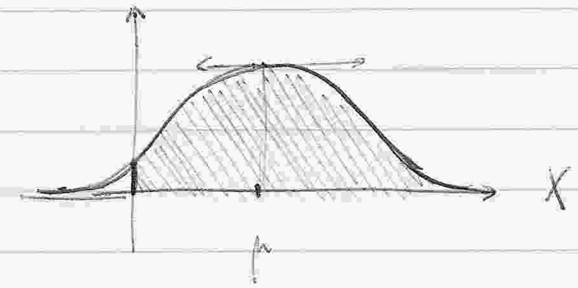
Variance dans la direction de

A, B : 2 classements $\qquad \Sigma$ = matrice de covariance $= \begin{pmatrix} v_A & c \\ c & v_B \end{pmatrix}$

$X = A - B \qquad \mu(X) = \mu(A) - \mu(B)$

σ^2, variance de $X = \begin{pmatrix} 1 & -1 \end{pmatrix} \Sigma \begin{pmatrix} 1 \\ -1 \end{pmatrix} = v_A + v_B - 2c$

$\mu, \sigma^2. \qquad P(X > 0) =$



$$I = \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$t = \frac{x-\mu}{\sqrt{2}\sigma} \qquad x = \mu + \sqrt{2}\sigma t \qquad dx = \sqrt{2}\sigma \, dt$$

$$I = \int_{-\frac{\mu}{\sqrt{2}\sigma}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2} dt = \int_{-\frac{\mu}{\sqrt{2}\sigma}}^{0} \frac{1}{\sqrt{\pi}} e^{-t^2} dt + \int_0^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2} dt$$

$$\boxed{\sigma^2 = v_A + v_B - 2c}$$

$$= \frac{1}{2}\operatorname{erf}(\infty) - \frac{1}{2}\operatorname{erf}\left(-\frac{\mu}{\sqrt{2}\sigma}\right)$$

$$\boxed{I = \frac{1}{2}\left(1 - \operatorname{erf}\left(-\frac{\mu}{\sqrt{2}\sigma}\right)\right)}$$

$$\boxed{I = \frac{1}{2}\operatorname{erfc}\left(-\frac{\mu}{\sqrt{2\sigma^2}}\right)}$$

Racine carrée de la matrice de covariance $K = (k_{ij})$

Programmation : tjs travailler avec la matrice réduite.

A, $A \cdot {}^tA = K$.

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} & & \\ & a_{22} & a_{32} & & \\ & & a_{33} & & \\ & 0 & & \ddots & \\ & & & & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & 0 & \\ a_{31} & a_{32} & a_{33} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} k_{11} & k_{12} & & & \\ k_{21} & k_{22} & & & \\ & & k_{33} & & \\ & & & \ddots & \\ & & & & k_{nn} \end{pmatrix}$$

$a_{11} = \sqrt{k_{11}}$

$a_{21} a_{11} = k_{21} \qquad a_{21} = \dfrac{k_{21}}{a_{11}}$

$\forall i > 1 \quad a_{i1} a_{11} = k_{i1} \qquad a_{i1} = \dfrac{k_{i1}}{a_{11}}$

$a_{21} a_{21} + a_{22} a_{22} = k_{22} \qquad a_{22} = \sqrt{k_{22} - a_{21}^2} \quad \cancel{= \sqrt{k_{22} - \dfrac{k_{21}^2}{k_{11}}}}$

$\forall i > 2 \quad a_{i1} a_{21} + a_{i2} a_{22} = k_{i2} \qquad a_{i2} = \dfrac{1}{a_{22}} \left(k_{i2} - a_{i1} a_{21} \right)$

$a_{31}^2 + a_{32}^2 + a_{33}^2 = k_{33} \qquad a_{33} = \sqrt{k_{33} - a_{31}^2 - a_{32}^2}$

$\forall i > 3 \quad a_{i1} a_{31} + a_{i2} a_{32} + a_{i3} a_{33} = k_{i3} \qquad a_{i3} = \dfrac{1}{a_{33}} \left(k_{i3} - a_{i1} a_{31} - a_{i2} a_{32} \right)$

etc.

Multi-Dimensional Elo Rating

n-dimensional

$r_i^{(k)}$ k-th component of the rating of player i

$\gamma_i^{(k)} = e^{r_i^{(k)}}$

$r_i^{(k)} = \ln \gamma_i^{(k)}$

r_d

r_w

$$\delta_{ij} = \frac{\sum_k \left(r_i^{(k)} - r_j^{(k)}\right)^3}{\sum_k \left(r_i^{(k)} - r_j^{(k)}\right)^2} + r_w \quad [i \text{ plays white}]$$

$$P(i \text{ beats } j) = \cancel{\frac{1}{1+e^{-\delta_{ij}+r_w}}} \quad \frac{1}{1+e^{-\delta_{ij}+r_d}}$$

$$P(j \text{ beats } i) = \cancel{\frac{1}{1+e^{\delta_{ji}}}} \quad \frac{1}{1+e^{+\delta_{ij}+r_d}}$$

$$P(i \text{ draws } j) = 1 - \frac{1}{1+e^{-\delta_{ij}+r_d}} - \frac{1}{1+e^{+\delta_{ij}+r_d}}$$

$$= \frac{\left(1+e^{-\delta_{ij}+r_d}\right)\left(1+e^{+\delta_{ij}+r_d}\right) - \left(1+e^{+\delta_{ij}+r_d}\right) - \left(1+e^{-\delta_{ij}+r_d}\right)}{\left(1+e^{-\delta_{ij}+r_d}\right)\left(1+e^{+\delta_{ij}+r_d}\right)}$$

$$= \frac{e^{2r_d} - 1}{(\quad)(\quad)} = \left(\gamma_d^2 - 1\right) P(i \text{ beats } j)\, P(j \text{ beats } i)$$

$$l_i = -\sum_j (w_{ij}+d_{ij}) \ln\left(1+e^{-\delta_{ij}+r_d}\right) + (l_{ij}+d_{ij}) \ln\left(1+e^{\delta_{ij}+r_d}\right) +$$

$$(w_{ji}+d_{ji}) \ln\left(1+e^{-\delta_{ji}+r_d}\right) + (l_{ji}+d_{ji}) \ln\left(1+e^{\delta_{ji}+r_d}\right)$$

$$\frac{\partial l_i}{\partial r_i^{(p)}} \; ? \qquad \frac{\partial}{\partial r_i^{(p)}} \ln\left(1+e^{-\delta_{ij}+r_d}\right) = \frac{e^{r_d} \frac{\partial}{\partial r_i^{(p)}}\left[e^{-\delta_{ij}}\right]}{1+e^{-\delta_{ij}+r_d}} = \frac{-e^{-\delta_{ij}+r_d}\left(\frac{\partial}{\partial r_i^{(p)}}\delta_{ij}\right)}{1+e^{-\delta_{ij}+r_d}}$$

$$= -\frac{\partial \delta_{ij}}{\partial r_i^{(p)}} \cdot \frac{1}{1+e^{\delta_{ij}-r_d}}$$

$$\frac{\partial \delta_{ij}}{\partial r_i^{(p)}} = \frac{3\left(r_i^{(p)} - r_j^{(p)}\right)^2 \sum_k \left(r_i^{(k)} - r_j^{(k)}\right)^2 - \sum_k \left(r_i^{(k)} - r_j^{(k)}\right)^3 \cdot 2\left(r_i^{(p)} - r_j^{(p)}\right)}{\left(\sum_k \left(r_i^{(k)} - r_j^{(k)}\right)^2\right)^3}$$

C'est aussi simple que ça ! $\quad \dfrac{\partial \delta_{ji}}{\partial r_i(p)} = - \dfrac{\partial \delta_{ij}}{\partial r_i(p)}$

$$\frac{\partial l_i}{\partial r_i^{(p)}} = \sum_j \left(w_{ij} + d_{ij}\right) \frac{\partial \delta_{ij}}{\partial r_i^{(p)}} \cdot \frac{1}{1 + e^{\delta_{ij} - r_d}} - \left(l_{ij} + d_{ij}\right) \frac{\partial \delta_{ij}}{\partial r_i^{(p)}} \cdot \frac{1}{1 + e^{-\delta_{ij} - r_d}}$$

$$\left(w_{ji} + d_{ji}\right) \frac{\partial \delta_{ji}}{\partial r_i^{(p)}} \cdot \frac{1}{1 + e^{\delta_{ji} - r_d}} - \left(l_{ji} + d_{ji}\right) \frac{\partial \delta_{ji}}{\partial r_i^{(p)}} \frac{1}{1 + e^{-\delta_{ji} - r_d}}$$